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LARGE-TIME BEHAVIOR OF SOLUTIONS TO A SCALAR CONSERVATION LAW IN SEVERAL SPACE DIMENSIONS

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ABSTRACT. We consider solutions of the Cauchy problem in $\mathbf{R}_+^{n+1}$ for the equation $u_t + \operatorname{div}_x f(u) = 0$. The initial data is assumed to be a compact perturbation of a function of the form, $\varphi(x) = a$ for $\langle x, \mu\rangle > 0$, $\varphi(x) = b$ for $\langle x, \mu\rangle < 0$, where a and b are constants and μ is a given unit vector. The Cauchy problem together with an entropy condition on u is known to be well posed. The solution with unperturbed initial data, $\varphi(x)$, is a traveling shock, $\varphi(x-\vec{k}t)$, provided that $\varphi(x-\vec{k}t)$ satisfies the entropy condition (an inequality on a, b, μ, and f). Assuming this type of condition on φ, we study the large-time behavior of u. In particular, we show that u converges to a traveling shock whose profile agrees with $\langle x, \mu\rangle = 0$ outside of a compact set.

Introduction. In 1958 A. M. Ilin and O. A. Oleinik [**4**] studied the large-time stability of a shock solution to a single conservation law. They considered the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{for } t > 0,\ x \in \mathbf{R}, \tag{0.1}$$

$$u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R}, \tag{0.2}$$

where u_0 is bounded,

$$u_0(x) = a \cdot \chi_{\mathbf{R}^+}(x) + b \cdot \chi_{\mathbf{R}^-}(x) \equiv \psi(x)$$

for $|x|$ sufficiently large,

$$a < b \quad \text{and} \quad f'' > 0. \tag{0.3}$$

Solutions of the Cauchy problem are unique among those which satisfy an entropy condition (see (E) in §1). The structure assumptions, (0.3), ensure that $\psi(x-kt)$ is a solution of (0.1) and satisfies the entropy condition when $k = (f(b)-f(a))/(b-a)$. Ilin and Oleinik used the viscosity method to show that for all t sufficiently large, $u(t,x) = \psi(x_0 + x - kt)$ for some $x_0 \in \mathbf{R}$. Thus if the initial data is a compact perturbation of ψ, the solution evolves after finite time into the traveling shock solution, $\psi(x_0+x-kt)$. We remark that the same result was proved by C. Dafermos [**2**] and T. P. Liu [**7**] using the notion of generalized characteristic curves.

In this paper we investigate the n-dimensional analog of this problem:

$$\partial u/\partial t + \operatorname{div} f(u) = 0 \quad \text{for } t > 0,\ x \in \mathbf{R}^n, \tag{0.4}$$

$$u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R}^n \tag{0.5}$$

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where $f \in C^1(\mathbf{R};\mathbf{R}^n)$ and $n \geq 1$. We assume that $u_0 \in L^\infty(\mathbf{R}^n)$ and $u_0 = \varphi$ for $|x| \geq R_1 > 0$, where

$$\varphi(x) = \begin{cases} a & \text{for } \langle x, \mu\rangle > 0, \\ b & \text{for } \langle x, \mu\rangle < 0 \end{cases}$$

for some unit vector $\mu \in \mathbf{R}^n$ and $a < b$. We define a solution of (0.4) to be a distribution solution in $C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$ which satisfies Vol'pert's entropy condition, (E). (See §1.)

If $k = [f(b) - f(a)]/(b-a)$, it follows from (E) that $\varphi(x - kt)$ is a solution of (0.4) if and only if $\langle \tilde{f}(c), \mu\rangle \leq 0$ for $a \leq c \leq b$ where $\tilde{f}(c) \equiv f(c) - f(a) - (c-a)k$. We assume throughout the paper that φ is *nondegenerate*, i.e., the above entropy condition holds in the following strict sense: there exists a constant $\theta > 0$ such that

$$\langle \mu, \tilde{f}(c)\rangle \leq -\theta(b-c)(c-a) \quad \text{for } a \leq c \leq b. \tag{0.6}$$

We summarize our results as follows. In §§1 and 2 we show that there are bounds $m_1 < a$ and $m_2 > b$ (determined explicitly by $\tilde{f}$) so that if

$$m_1 < \text{ess inf}\, u_0 \quad \text{and} \quad m_2 > \text{ess sup}\, u_0, \tag{0.7}$$

then after a finite time, $a \leq u(t,x) \leq b$. Moreover, there is a set $M \subset \mathbf{R}^n$ and a constant $R_2 > 0$ such that if

$$v(x) = b \cdot \chi_M(X) + a \cdot \chi_{\mathbf{R}^n - M}(x),$$

then

$$u(t,x) - v(x-kt) \to 0 \quad \text{in } L^1(\mathbf{R}^n) \text{ as } t \to \infty$$

and

$$M \cap \{x : |x| > R_2\} = \{x : \langle x, \mu\rangle < 0,\ |x| > R_2\}.$$

Our investigation shows that the large-time behavior of u can be described more definitely and that it depends on $d \equiv$ dimension (span$\{\tilde{f}(c) : a \leq c \leq b\}$). In §3 we study the case $d = n$ and prove the following results. Here the initial data is called *admissible* if it satisfies (0.7) and $u_0 = \varphi$ for $|x| \geq R_1 > 0$.

THEOREM 3.2. *If u_0 is admissible and $d = n$, then M is a Lipschitz domain. More precisely, there is a set of coordinates $(y_1, \ldots, y_n) = (\hat{y}, y_n)$ obtained by a rotation and a constant, C, (both depending only on $\tilde{f}$) so that $M = \{(\hat{y}, y_n) : y_n < g(\hat{y})\}$ where g is Lipschitz continuous and $\|\nabla g\|_\infty \leq C$.*[3]

We prove the following stability estimate for the shock front, g.

THEOREM 3.3. *Suppose u_1 and u_2 are admissible initial data and $d = n$. Let g_1 and g_2 be the end state shock fronts. Then*

$$\sup_{\hat{y} \in \mathbf{R}^{n-1}} |g_1(\hat{y}) - g_2(\hat{y})| \leq C(\tilde{f}) \cdot \left(\int_{\mathbf{R}^n} |u_1 - u_2| dx \right)^{1/n}.$$

Using the above estimate, we show that in finite time, u is equal to the traveling shock, $v(x-kt)$, outside an arbitrarily small neighborhood of the shock front.

[3]For $n = 1$, g should be replaced by a constant, x_0, and $M = \{x : x < x_0\}$.

THEOREM 3.12. *Suppose $d = n$ and u_0 is admissible and piecewise continuous. (See 3.5.) If $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $u(t,x) = v(x - kt)$ for $t \geq T_\varepsilon$ and* $\operatorname{dist}(x - kt, \partial M) > \varepsilon$.

In §4 we study the case $d < n$ and show that it reduces to the case described above. Without loss of generality we assume that $V \equiv \operatorname{span}\{\tilde{f}(c): a \leq c \leq b\} = \mathbf{R}^d \times \{0\}$; the projection of μ on V is nontrivial; and its unit-length normalization, $(\mu', 0)$, satisfies

$$\langle(\mu', 0), \tilde{f}(c)\rangle \leq -\theta(b - c)(c - a) \quad \text{for } a \leq c \leq b.$$

(This holds after an appropriate change of coordinates.)

The problem, (0.4) and (0.5), can then be interpreted (after a finite time) as a parametrized set of problems in d space dimensions. To describe this in more detail, let us denote $(x_1, \ldots, x_n)$ by (x', x'') with $x' \in \mathbf{R}^d$ and $x'' \in \mathbf{R}^{n-d}$. We prove

LEMMA 4.1. *Suppose u_0 is admissible and $d < n$. Under the above hypotheses on $\tilde{f}$, there exists $T > 0$ (explicitly computable from $\tilde{f}$) so that if $t \geq T$,*

$$u(t,x) = \tilde{u}(t, x' - k't; x'' - k''t).$$

Here $\tilde{u}(t, x'; x'')$ is defined for a.e. $x'' \in \mathbf{R}^{n-d}$ as the solution of the problem

$$\frac{\partial \tilde{u}}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(\tilde{f}_i(\tilde{u})) = 0 \quad \textit{for } t \geq T;\ x' \in \mathbf{R}^d,$$

$$\tilde{u}(T, x'; x'') = u(T, x' + k'T, x'' + k''T).$$

The function $u(T, x' + k'T, x'' + k''T)$ for fixed $x'' \in \mathbf{R}^{n-d}$ is admissible relative to a nondegenerate shock defined through μ'.

From our results for the case $d = n$, we have

THEOREM 4.6. *Suppose u_0 is admissible, piecewise continuous, and $d < n$. For all $\varepsilon > 0$ and a.e. $z'' \in \mathbf{R}^{n-d}$ there exists $T = T(\varepsilon, z'') > 0$ and a Lipschitz domain, $M(z'')$, in $\mathbf{R}^d$ such that $\chi_M(z', z'') = \chi_{M(z'')}(z')$ and if $x'' - k''t = z''$,*

$$\begin{aligned} u(t, x', x'') &= v(t, x' - k't, x'' - k''t) \\ &= b \cdot \chi_{M(z'')}(x' - k't) + a \cdot \chi_{\mathbf{R}^d - M(z'')}(x' - k't) \end{aligned}$$

for $t \geq T$ and $\operatorname{dist}(x' - k't, \partial M(z'')) \geq \varepsilon$.

Solutions of (0.4) and (0.5) when $n = 2$ and u_0 is constant in each of the four quadrants were studied by D. Wagner in [**9**]. He assumed that $f = (f_1, f_2)$ with f_1'', $f_2'' > 0$ and f_1 sufficiently close to f_2. Under these assumptions he constructed the solution and analyzed its qualitative properties. Further study of this problem was done by W. B. Lindquist [**6**] assuming that $f_1 = f_2$ and f_1 has at most two inflection points.

In [**1**] E. Conway studied the large-time behavior of solutions of (0.4) and (0.5) in n space dimensions assuming that u_0 has compact support and f is convex in some direction. He proved decay estimates on the L^∞-norm of such solutions.

As far as we know, all previous studies of the large-time behavior of u assumed some type of convexity condition on f. Our primary assumption is (0.6). In particular, our results are new even for the case $n = 1$.

1. Super and subentropy solutions. In this section we define a notion of super and subentropy solutions of a conservation law. A technique used throughout the paper is the explicit construction of such functions to obtain estimates on $u(t,x)$ for large time.

First, let us recall some results proved by Vol'pert in [**8**]. Suppose $u \in BV(\mathbf{R}_+^{n+1}) \cap L^\infty(\mathbf{R}_+^{n+1})$. Then Vol'pert showed that with the exception of a set of H^n-measure zero every (t,x) in $\mathbf{R}^{n+1}$ is a *regular point* of u. This means that there is a unit vector, ν, in $\mathbf{R}^{n+1}$ and numbers $l_\nu u(t,x)$ and $l_{-\nu}u(t,x)$ such that for each $\varepsilon > 0$

$$\lim_{r\to 0} \frac{|\{(\tau,y)\in B_r(t,x): |u(\tau,y) - l_{\pm\nu}u(t,x)| > \varepsilon \text{ and } \langle(\tau - t, y - x), \pm\nu\rangle \geq 0\}|}{|B_r(t,x)|} = 0.$$

(Here $|E|$ denotes the Lebesgue measure of a set, E.) We denote the set of regular points where $l_\nu u(t,x) \neq l_{-\nu}u(t,x)$ as the set of approximate jump points, $\Gamma(u)$. We remark that up to a multiple of ± 1, ν is unique for $(t,x)\in\Gamma(u)$ and is called a normal to $\Gamma(u)$ at (t,x). We adopt the convention that $u^+ = l_\nu u(t,x)$, $u^- = l_{-\nu}u(t,x)$, and ν is chosen so that $u_+ \leq u_-$. If $g(t)$ is a continuously differentiable function, Vol'pert proved that $g(u(t,x)) \in BV(\mathbf{R}_+^{n+1})$ and $\nabla g(u) = \tilde{g}(u)\cdot\nabla u$ as a Borel measure, where for each regular point, (t,x),

$$\tilde{g}(u) = \begin{cases} \dfrac{g(u^+) - g(u^-)}{u^+ - u^-} & \text{if } (t,x)\in\Gamma(u), \\ g'(\bar{u}(t,x)) & \text{if } (t,x)\notin\Gamma(u), \end{cases}$$

and $\bar{u}(t,x) = \lim_{r\to 0} ⨍_{B_r(t,x)} u(\tau,y)\,d\tau\,dy$, the Lebesgue limit of u at (t,x). (This is well defined H^n-almost everywhere.)

Vol'pert proved that solutions of (0.4) and (0.5) are unique among those which satisfy an entropy condition. More precisely, suppose $u_0 \in BV(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$. Let $A(u) = (u, f(u))$ and define $S: \mathbf{R}^2 \to \mathbf{R}^{n+1}$ by

$$S(u,v) = (A(u) - A(v))\cdot \operatorname{sign}(u - v).$$

Then there is a unique function $u \in BV(\mathbf{R}_+^{n+1})$ satisfying

$$\partial u/\partial t + \operatorname{div} f(u) = \operatorname{div} A(u) = 0, \tag{1.1}$$

$$\lim_{t\to 0+} \bar{u}(t,x) = u_0(x) \quad \text{for } L^n\text{-a.e. } x \tag{1.2}$$

and the entropy condition:

(E) $\qquad \operatorname{div} S(u,c) \leq 0$ for all $c \in \mathbf{R}$ as a distribution on $\mathbf{R}_+^{n+1}$.

When $u \in BV(\mathbf{R}_+^{n+1}) \cap L^\infty(\mathbf{R}_+^{n+1})$ and u satisfies (1.1), the entropy condition, (E), is equivalent to

(E′) $\quad \langle S(u^+,c),\nu\rangle \leq \langle S(u^-,c),\nu\rangle$ for all $c\in\mathbf{R}$ and H^n-almost every point in the jump set, $\Gamma(u)$.

In addition, Vol'pert proved that if $u_0 \in BV(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$, then

(i) $\bar{u}(t,x) \in C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$,

(ii) $|\bar{u}| \leq \|u_0\|_\infty$,

(iii) For any two solutions, u and w, of (1.1) which satisfy (E) and such that u, $w \in L^\infty(\mathbf{R}_+^{n+1}) \cap C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$ and $|u|$, $|w| \leq m < \infty$, we have

$$\int_{B_r} |\bar{u}(t,x) - \bar{w}(t,x)|\,dx \leq \int_{B_{r+lt}} |u(x,0) - w(x,0)|\,dx \tag{1.3}$$

where $l = \left(\sum_{i=1}^n (\max_{|u|\leq m} |f_i'(u)|)^2\right)^{1/2}$.

If $u_0 \in L^\infty(\mathbf{R}^n)$, Vol'pert showed that there is a distribution solution of (1.1) and (1.2) in the class $C([0,\infty); L^1_{\mathrm{loc}}(\mathbf{R}^n))$ which satisfies (E) and also satisfies (ii) and (iii). Kruzkov proved that it is unique in a broader class [**5**].

As stated in the introduction, we define a *solution* of (1.1) to be a function, u, in $C([0,\infty); L^1_{\mathrm{loc}}(\mathbf{R}^n))$ which satisfies (1.1) in the sense of distributions and also satisfies the entropy condition (E).

We define super and subentropy solutions as follows:

DEFINITION 1.4. Suppose

$$u(t,x) \in BV(\mathbf{R}^{n+1}_+) \cap L^\infty(\mathbf{R}^{n+1}_+) \cap C([0,\infty); L^1_{\mathrm{loc}}(\mathbf{R}^n)).$$

Then u is a *super* (*sub*) *solution* of (1.1) if

(1) $$\operatorname{div} A(u) \equiv \langle [A(u^+) - A(u^-)], \nu\rangle d\mathcal{H}^n|_{\Gamma(u)} + \left(u_t + \sum_{i=1}^n f_i'(u)\cdot u_{x_i}\right)\Bigg|_{\Gamma(u)^c} \geq 0\ (\leq 0).$$

(2) For $\mathcal{H}^n$-a.e. $(t,x) \in \Gamma(u)$,

$$\langle A(u^+), \nu\rangle \geq (A(c), \nu) \quad \text{for } u^+ \leq c \leq u^-$$
$$(\text{sub: } \langle A(u^-), \nu\rangle \geq \langle A(c), \nu\rangle \text{ for } u^+ \leq c \leq u^-).$$

The inequalities in condition (2) are splitting of the entropy condition into two inequalities.

The following lemma will be used to prove a maximum principle for super and subsolutions. We use the notation $\sigma^+(z) = \chi_{\mathbf{R}^+}(z)$, $\sigma^-(z) = -\chi_{\mathbf{R}^-}(z)$, and $S^\pm(u,v) = [A(u) - A(v)]\cdot \sigma^\pm(u-v)$.

LEMMA 1.5. *Suppose* $u \in BV(\mathbf{R}^{n+1}_+) \cap L^\infty(\mathbf{R}^{n+1}_+) \cap C([0,\infty); L^1_{\mathrm{loc}}(\mathbf{R}^n))$. *Then* u *is a supersolution iff* $\operatorname{div} S^-(u,c) \leq 0$ *for all* $c \in \mathbf{R}$; u *is a subsolution iff* $\operatorname{div} S^+(u,c) \leq 0$ *for all* $c \in \mathbf{R}$.

PROOF. Let $\Delta S^-(u,c) = [S^-(u^+,c) - S^-(u^-,c)]$ in $\Gamma(u)$. Then from the calculus of BV functions [**8**, §15] we have

$$\operatorname{div} S^-(u,c) = \overline{\sigma^-}(u-c)\cdot \operatorname{div} A(u)|_{\Gamma(u)^c} + \langle \Delta S^-(u,c), \nu\rangle d\mathcal{H}^n|_{\Gamma(u)}.$$

Note first that $\langle \Delta S^-(u,c), \nu\rangle \leq 0$ a.e. on $\Gamma(u)$ for $c \in \mathbf{R}$ iff $\langle A(c), \nu\rangle \leq \langle A(u^+), \nu\rangle$ for $u^+ \leq c \leq u^-$ a.e. on $\Gamma(u)$. Second, if we choose $c > ||u||_\infty$, then $\overline{\sigma^-}(u-c) \equiv -1$. Thus

$$\overline{\sigma^-}(u-c)\cdot \operatorname{div} A(u)|_{\Gamma(u)^c} \leq 0 \quad \text{for all } c \in \mathbf{R}$$

is equivalent to $\operatorname{div} A(u) \geq 0$ in $\Gamma(u)^c$. This shows that $\operatorname{div} S^-(u,c) \leq 0$ iff u is a supersolution. The result for subsolutions is proved similarly.

THEOREM 1.6. *Suppose* u *is a supersolution and* v *is a subsolution. Let*

$$m = \sup\{\max\{|u(t,x)|, |v(t,x)|\} : x \in \mathbf{R}^n, 0 \leq t \leq T\}$$

and

$$l = \left(\sum_{i=1}^n \left(\max_{|u| \leq m} |f_i'(u)| \right)^2 \right)^{1/2}.$$

Then $\operatorname{div} S^-(u,v) = \operatorname{div} S^+(v,u) \leq 0$ *and for* $0 \leq t \leq T < \infty$,

$$\int_{B_r} \max\{0, v-u\}(t,x)\,dx \leq \int_{B_{r+lt}} \max\{0, v-u\}(0,x)\,dx.$$

PROOF. We have from [**8**, §15] that

$$\operatorname{div} S^-(u,v) = \overline{\sigma^-}(u-v) \cdot \operatorname{div}[A(u) - A(v)]|_{\Gamma(u-v)^c} + \langle \Delta S^-(u,v), \nu\rangle d\mathcal{H}^n|_{\Gamma(u-v)}$$

where $\Delta S^-(u,v) = S^-(l_\nu u, l_\nu v) - S^-(l_{-\nu}u, l_{-\nu}v)$ and ν is the normal to $\Gamma(u-v)$. The first term is a nonpositive measure. As for the second term, it follows from Lemma 1.5 that for any c

(1) $\langle S^-(l_\nu u, c), \nu\rangle \leq \langle S^-(l_{-\nu}u, c), \nu\rangle$ and

(2) $\langle S^+(l_\nu v, c), \nu\rangle \leq \langle S^+(l_{-\nu}v, c), \nu\rangle$,

$\mathcal{H}^n$-almost everywhere on $\Gamma(u-v)$. Set $c = l_\nu v$ in (1) and $c = l_{-\nu}u$ in (2). Since $S^+(h,g) = S^-(g,h)$, it follows from (1) and (2) that $\Delta S^-(u,v) \leq 0$ a.e. on $\Gamma(u-v)$. This proves that $\operatorname{div} S^-(u,v) = \operatorname{div} S^+(v,u) \leq 0$.

The second assertion follows from Green's theorem on the region, $D = \{(x,\tau)\colon |x| < r + l(t-\tau), 0 < \tau < t\}$. We have (from [**8**, §14])

$$\begin{aligned} 0 \geq \int_D \operatorname{div} S^-(u,v)\,d\tau\,dx &= \int_{B_r} \max\{0, v-u\}(t,x)\,dx \\ &- \int_{B_{r+lt}} \max\{0, v-u\}(0,x)\,dx \\ &+ \int_{\partial D \cap \{0<\tau<t\}} \langle S^-(l(u), l(v)), N\rangle d\mathcal{H}^n \end{aligned}$$

where $l(u)$, $l(v)$ are the inward traces of u, v and N is the outward-pointing normal on ∂D. From the choice of l the third term is nonnegative. This proves the theorem.

REMARK 1.7. The above theorem also holds if $v \notin BV(\mathbf{R}_+^{n+1})$ but $v \in L^\infty(\mathbf{R}_+^{n+1}) \cap C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$ and v is a solution of (1.1). This follows from Vol'pert's construction of v as a limit in $C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$ of a sequence of functions, $\{v_j(t,x)\}$, which are solutions of (1.1) and satisfy the hypotheses of Theorem 1.6.

2. Preliminary results. In this section we prove preliminary results concerning the large-time behavior of solutions of (1.1) and (1.2). We assume throughout the paper that $f \in C^1(\mathbf{R}; \mathbf{R}^n)$ for $n \geq 1$, $u(t,x)$ is a solution of (1.1) and (1.2), and u_0 is a compact perturbation of the Riemann data,

$$\varphi(x) = \begin{cases} a & \text{for } \langle x, \mu\rangle > 0, \\ b & \text{for } \langle x, \mu\rangle < 0, \end{cases}$$

where μ is a unit vector in $\mathbf{R}^n$ and $a < b$.

We define

$$k = [f(b) - f(a)]/(b-a) \quad \text{and} \quad \tilde{f}(c) = f(c) - f(a) - (c-a)k.$$

Our main assumption (involving f and φ) is that φ is *nondegenerate*; that is, there exists a constant $\theta > 0$ such that

$$\langle \tilde{f}(c), \mu\rangle \leq -\theta(b-c)(c-a) \quad \text{for } a \leq c \leq b. \tag{2.1}$$

This is essentially a strict entropy condition on $\varphi(x-kt)$. To check this, we note that $w(t,x) \equiv \varphi(x-kt)$ is in $BV(\mathbf{R}_+^{n+1}) \cap L^\infty(\mathbf{R}_+^{n+1})$ and is a weak solution of (1.1). The entropy condition (E) on w is equivalent to (E′) which can be stated in the form

$$\text{(E}''\text{)} \qquad \langle A(c) - A(w^+), \nu\rangle \leq \langle A(w^-) - A(c), \nu\rangle \quad \text{for } w^+ \leq c \leq w^-$$

H^n-a.e. in $\Gamma(w)$. (See §1.) Since ν is in the same direction as $(\langle -k, \mu\rangle, \mu)$ and $\tilde{f}(a) = \tilde{f}(b) = 0$, (E″) reduces to

$$\langle \tilde{f}(c), \mu\rangle \leq 0 \quad \text{for } a \leq c \leq b.$$

Thus (2.1) ensures that $\varphi(x-kt)$ is a traveling wave solution of (1.1).

We define

$$m_1 = \inf\left\{ m : m < a \text{ and } \alpha_1(m) \equiv \max_{m\leq c\leq a} \frac{\langle \tilde{f}(c), \mu\rangle}{c-a} < \beta_1(m) \equiv \min_{m\leq c\leq a} \frac{\langle \tilde{f}(c), \mu\rangle}{c-b} \right\}$$

and

$$m_2 = \sup\left\{ m : m > b \text{ and } \alpha_2(m) \equiv \max_{b\leq c\leq m} \frac{\langle \tilde{f}(c), \mu\rangle}{c-a} < \beta_2(m) \equiv \min_{b\leq c\leq m} \frac{\langle \tilde{f}(c), \mu\rangle}{c-b} \right\}.$$

By (2.1) and the differentiability of f, m_1 and m_2 are well defined with $-\infty \leq m_1 < a$ and $b < m_2 \leq +\infty$. The initial data, u_0, is called *admissible* if

(i) for some $R_1 > 0$, $u_0(x) = \varphi(x)$ for $|x| \geq R_1$, and

(ii) $u_0 \in L^\infty(\mathbf{R}^n)$, $m_1 < \operatorname{ess\,inf} u_0$, and $m_2 > \operatorname{ess\,sup} u_0$.

We first prove that if u_0 is admissible then for some $t^* < \infty$, we have $a \leq u(t,x) \leq b$ for $t \geq t^*$. It will be convenient in the proof to assume (without loss of generality) that $f(a) = f(b) = 0$ and hence $k = 0$. This is possible because if $k \neq 0$ we can make a change of variables,

$$\text{(2.2)} \qquad x_1 = x - kt, \quad t_1 = t,$$

and set $\tilde{u}(t_1, x_1) = u(t,x)$. Equations (1.1), (1.2), and the entropy condition (E) become

$$\partial\tilde{u}/\partial t_1 + \operatorname{div}_{x_1} \tilde{f}(\tilde{u}) = 0, \quad \tilde{u}(0,x) = u_0(x),$$

and

$$(\tilde{\text{E}}) \qquad \operatorname{div}_{(t_1,x_1)} \tilde{S}(\tilde{u}, c) \leq 0 \quad \text{for all } c \in \mathbf{R}$$

where $\tilde{S}(v,w) = \operatorname{sign}(v-w)\cdot[\tilde{A}(v) - \tilde{A}(w)]$ and $\tilde{A}(v) = (v, \tilde{f}(v))$. Thus the effect is to replace f by $\tilde{f}$ which satisfies $\tilde{f}(a) = \tilde{f}(b) = 0$.

THEOREM 2.3. *If u_0 is admissible, there exists $t^* < \infty$ (defined below) such that $a \leq u(t,x) \leq b$ for $t \geq t^*$.*

PROOF. Let $b' = \operatorname{ess\,sup} u_0$ and $a' = \operatorname{ess\,inf} u_0$. Define

$$t_1 = \begin{cases} 0 & \text{if } a' \geq a, \\ 2R_1/[\beta_1(a') - \alpha_1(a')] & \text{if } a' < a, \end{cases}$$

$$t_2 = \begin{cases} 0 & \text{if } b' \leq b, \\ 2R_1/[\beta_2(b') - \alpha_2(b')] & \text{if } b' > b, \end{cases}$$

and let $t^* = \max\{t_1, t_2\}$.

First we show that $u \le b$ for $t \ge t^*$. We assume that $b' > b$ because if not, $u \le b$ for all time by the maximum principle. We also assume (w.l.o.g.) that $f(a) = f(b) = 0$ and hence $f = \tilde{f}$.

Let $\alpha = \alpha_2(b')$, $\beta = \beta_2(b')$, and consider the function defined by

$$w(t,x) = \begin{cases} b & \text{for } \langle x, \mu\rangle < \beta t - R_1, \\ b' & \text{for } \beta t - R_1 < \langle x, \mu\rangle < \alpha t + R_1, \\ a & \text{for } \alpha t + R_1 < \langle x, \mu\rangle, \end{cases}$$

when $0 \le t \le 2R_1/(\beta - \alpha)$;

$$w(t,x) = \begin{cases} b & \text{for } \langle x, \mu\rangle < 2R_1\alpha/(\beta-\alpha) + R_1, \\ a & \text{for } 2R_1\alpha/(\beta-\alpha) + R_1 < \langle x, \mu\rangle, \end{cases}$$

when $t > 2R_1/(\beta - \alpha)$.

We claim that w is a supersolution. Clearly $w \in BV(\mathbf{R}_+^{n+1}) \cap L^\infty(\mathbf{R}_+^{n+1}) \cap C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$ and since w is piecewise constant we need only verify (1.4.2) on $\Gamma(w)$.

On the jump from $w = b$ to $w = b'$ for $t \le 2R_1/(\beta - \alpha)$, we have $\nu = (\beta, -\mu)/\sqrt{1+\beta^2}$. We need

$$\langle A(b) - A(c), \nu\rangle \ge 0 \quad \text{for } b \le c \le b'.$$

The left-hand side has the same sign as $-(c-b)\beta + \langle f(c), \mu\rangle$ and this is nonnegative for $b \le c \le b'$ by the definition of β. On the jump from $w = b'$ to $w = a$, we take ν in the direction of $(-\alpha, \mu)$. It is required that $\langle A(a) - A(c), \nu\rangle \ge 0$ for $a \le c \le b'$. This is equivalent to

$$\alpha(c-a) - \langle f(c), \mu\rangle \ge 0 \quad \text{for } a \le c \le b'$$

which holds by (2.1) and the definition of α since $\alpha \ge \langle f(b), \mu\rangle/(b-a) = 0$.

The entropy condition holds across the jump from $w = b$ to $w = a$ for $t > 2R_1/(\beta - \alpha)$ since $\nu = (0, \mu)$ and thus the entropy condition (E′) reduces to $\langle f(c), \mu\rangle \le 0$ for $a \le c \le b$. By Remark 1.7, we have

$$\int_{B_r} \max\{0, w-u\}(t,x)\,dx \le \int_{B_{r+lt}} \max\{0, w-u\}(0,x)dx = 0$$

for any t and $r \ge 0$ where $l = \left[\sum_{i=1}^n (\max_{|u|\le|b'|+|a'|} |f_i'(u)|)^2\right]^{1/2}$. We conclude that $u \le w$ for all t and a.e. $x \in \mathbf{R}^n$, and hence $u \le b$ for $t \ge t^*$ and a.e. $x \in \mathbf{R}^n$. To show that $u \ge a$ when $t \ge t^*$ involves the construction of a subsolution in a similar way.

A consequence of Theorem 2.3 is the following result on the large-time behavior of u.

THEOREM 2.4. *Suppose u_0 is admissible. There exists $R_3 > 0$ and a set $M \subset \mathbf{R}^n$ such that if $v(x) = b \cdot \chi_M(x) + a \cdot \chi_{\mathbf{R}^n - M}(x)$, then*

(i) *$u(t,x) - v(x - kt) \to 0$ in $L^1(\mathbf{R}^n)$ as $t \to \infty$,*

(ii) *$u(t,x) = v(x-kt)$ for $t > 0$ and $|x - kt| > R_3$, and*

(iii) *$M \cap \{x \in \mathbf{R}^n : |x| > R_3\} = \{x \in \mathbf{R}^n : \langle x, \mu\rangle < 0$ and $|x| > R_3\}$.*

To prove this, we will need the following lemma.

LEMMA 2.5. *Suppose u_0 is admissible. There exists $R_2 > R_1$ such that $u(t,x) = \varphi(x-kt)$ if $t \geq 0$ and $|x-kt| \geq R_2$.*

PROOF. By the change of variables (2.2), it is sufficient to prove the lemma assuming $f(a) = f(b) = 0$, so that $k = 0$ and $f = \tilde{f}$. Since φ is then a steady state solution of (1.1) and $u_0 = \varphi$ for $|x| \geq R_1$, it follows from estimate (1.3) that for some $r_0 > R_1$,

$$u(t^*, x) = \varphi(x) \quad \text{for } |x| \geq r_0$$

where t^* is as defined in Theorem 2.3. We will show that in fact there exists $r_1 > r_0$ such that

$$u(t,x) = \varphi(x) \quad \text{for } t \geq t^*,\ |x| \geq r_1.$$

To prove this we use the nondegeneracy condition, (2.1). It ensures that for some small $\delta > 0$, $\langle f(c), \nu\rangle \leq 0$ for $a \leq c \leq b$ and all $\nu \in \mathbf{R}^n$ such that $A(\mu,\nu) < \delta$, where $A(\mu,\nu)$ is the angle between μ and ν that is not greater than π. Hence we can construct open sets, M_1 and M_2 (with smooth boundaries) and choose $r_1 > r_0$ such that

$$\begin{aligned}
&M_1 \subset \{x : \langle x, \mu\rangle < 0\} \subset M_2,\\
&B_{r_0} \equiv B_{r_0}(0) \subset M_2 - M_1,\\
&M_i \cap \{x : |x| \geq r_1\} = \{x : \langle x, \mu\rangle < 0 \text{ and } |x| \geq r_1\} \quad \text{for } i = 1, 2, \text{ and}\\
&\langle f(c), \nu\rangle \leq 0 \quad \text{for } a \leq c \leq b,
\end{aligned}$$

where ν is any outward pointing normal to M_1 or M_2. As a result the functions, $v_i(x) = b \cdot \chi_{M_i}(x) + a \cdot \chi_{\mathbf{R}^n - M_i}(x)$, are steady state solutions of (1.1). Since $v_1(x) \leq u(t^*, x) \leq v_2(x)$ for all x, we have

$$v_1(x) \leq u(t,x) \leq v_2(x) \quad \text{for } x \in \mathbf{R}^n,\ t \geq t^*.$$

We conclude that

$$u(t,x) = \varphi(x) \quad \text{for } |x| \geq r_1,\ t \geq t^*.$$

By (1.3) the above equation holds when $t \geq 0$ and $|x| \geq R_2 \equiv r_1 + lt^*$ which proves the conclusion of the lemma.

PROOF OF THEOREM 2.4. As before, we may assume that $f(a) = f(b) = 0$ and hence $k = 0$.

Suppose $u_0 \in BV(\mathbf{R}^n)$. Consider $u_h(t,x) = u(t, x + he_i)$ for any $i = 1, \ldots, n$ and $|h| \leq R_1$, where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbf{R}^n$. Since u_h is a solution of (1.1), we have from Vol′pert's results (or Theorem 1.6) that $\operatorname{div} S(u_h, u) \leq 0$. Let $R = 3 \cdot R_2$ with R_2 as defined in Lemma 2.5 and $B_R = B_R(0)$. By applying Green's theorem in the cylinder, $(0,t) \times B_R$, we obtain

$$\begin{aligned}
\int_{B_R} |u(t, x+he_i) - u(t,x)|\,dx &\leq \int_{B_R} |u(0, x+he_i) - u(0,x)|\,dx\\
&\quad - \int_0^t \int_{\partial B_R} \left\langle \frac{x}{R}, ([f(u_h) - f(u)] \cdot \operatorname{sign}(u_h - u))\right\rangle d\mathcal{H}^n.
\end{aligned}$$

By Lemma 2.5, $f(u_h) = f(u) = 0$ on ∂B_R for any t. Hence the third term is zero and

$$\int_{B_R} \left|\frac{\partial u}{\partial x_i}(t,x)\right| dx \leq \int_{B_R} \left|\frac{\partial u}{\partial x_i}(0,x)\right| dx.$$

This and equation (1.1) imply that

$$\int_{-1}^{1}\int_{B_R}|(Du)(s+t,x)|dx\,ds \le c\cdot\int_{B_R}|(\nabla_x u)(0,x)|dx$$

for any $t>1$, where $D=(\partial/\partial t, \partial/\partial x_1,\dots,\partial/\partial x_n)$.

Now multiply equation (1.1) by $\langle x,\mu\rangle$ and integrate over the cylinder $(t^*,T)\times B_R$. We obtain

$$\int_{B_R}[u(T,x)-u(0,x)]\cdot\langle x,\mu\rangle dx$$
$$=\int_{t^*}^{T}\int_{B_R}\langle f(u),\mu\rangle dx\,dt-\int_{\partial B_R}\left\langle f(u),\frac{x}{R}\right\rangle\cdot\langle x,\mu\rangle d\mathcal{H}^n.$$

The first term is bounded uniformly in T (by estimate (1.3)) and the last term is zero. Since $a\le u\le b$ for $t\ge t^*$, we conclude using (2.1) that

$$\int_0^\infty\int_{B_R}|(u-a)(u-b)|dx\,dt<\infty.$$

By Lemma 2.5, this can be restated as

$$\int_{\mathbf{R}_+^{n+1}}|(u-a)(u-b)|dx\,dt<\infty.$$

From the above estimates (and Lemma 2.5) we see that for some sequence, $t_j\to+\infty$, $\{u(s+t_j,x)\}$ converges in $L^1_{\text{loc}}((-1,1)\times\mathbf{R}^n)$ to a function, $v(s,x)\in BV((-1,1)\times\mathbf{R}^n)$. The L^1 bound on $(u-a)(u-b)$ implies that $(v-a)(v-b)=0$ pointwise almost everywhere in $(-1,1)\times\mathbf{R}^n$. Thus

$$v(s,x)=b\cdot\chi_M(s,x)+a\cdot\chi_{\mathbf{R}^n-M}(s,x)$$

with $\chi_M\in BV((-1,1)\times\mathbf{R}^n)$. On the other hand, $\operatorname{div}S(v,c)\le 0$ as a measure on $(-1,1)\times\mathbf{R}^n$ for all $c\in\mathbf{R}$. Hence v is a solution of (1.1) on $(-1,1)\times\mathbf{R}^n$. (See [**8**, §16].) Since $f(a)=f(b)=0$, we have

$$0=\partial v/\partial s+\operatorname{div}f(v)=\partial v/\partial s \quad \text{on } (-1,1)\times\mathbf{R}^n.$$

Thus $\chi_M(s,x)=\chi_M(x)$.

Finally we show that $u(t,x)-v(x)\to 0$ in $L^1(\mathbf{R}^n)$ as $t\to\infty$. By Lemma 2.5 it is sufficient to prove convergence in $L^1(B_R)$. This follows from estimate (1.3) which implies that

$$\int_{B_R(0)}|v(x)-u(t,x)|dx\le\int_{B_R}|v(x)-u(s+t_j,x)|dx$$

for $t\ge s+t_j$. This proves (i) when $u_0\in BV(\mathbf{R}^n)$; (ii) and (iii) follow from (i) and Lemma 2.5 with $R_3=R=3\cdot R_2$.

For the case when $u_0\notin BV(\mathbf{R}^n)$ we can approximate u_0 by admissible functions, u_j, in $BV(\mathbf{R}^n)$ such that $u_j\to u_0$ in $L^1(B_{R_3})$ and $u_j(x)=\varphi(x)$ for $|x|\ge R_1$. By Lemma 2.5 if $u_j(t,x)$ is the solution of (1.1) with initial values, $u_j(x)$, then

$$u_j(t,x)=u(t,x)=\varphi(x)\quad\text{for } |x|\ge R_2.$$

By estimate (1.3) it follows that

$$\int_{B_{R_3}} |v_i(x) - v_j(x)|dx \leq \int_{B_{R_3}} |u_i(x) - u_j(x)|dx$$

where $v_j(x) = \lim_{t\to\infty} u_j(t,x)$. Hence for some set M, $\chi_{M_j} - \chi_M \to 0$ in $L^1(\mathbf{R}^n)$ and so

$$u(t,x) - b \cdot \chi_M(x) - a \cdot \chi_{\mathbf{R}^n - M}(x) \to 0$$

in $L^1(\mathbf{R}^n)$ as $t \to \infty$.

3. The n-dimensional case. We now get a more precise description of the set M and the convergence of u to $v(x-kt)$ in the special case

$$n = d \equiv \dim \operatorname{span}\{\tilde{f}(c) : a \leq x \leq b\}. \tag{3.1}$$

THEOREM 3.2. *There is a system of coordinates, $(\hat{y}, y_n)$, obtained by a rotation and a constant C_0 (both depending only on $\tilde{f}$) so that if u_0 is admissible then $M = \{(\hat{y}, y_n): y_n < g(\hat{y})\}$. The shock front, g, is Lipschitz continuous with Lipschitz constant at most C_0.*

PROOF. First assume $u_0 \in BV(\mathbf{R}^n)$ and w.l.o.g. $f(a) = f(b) = 0$ so that $f = \tilde{f}$. From the proof of Theorem 2.4, $v(x)$ is a steady state solution of (1.1) and $v \in BV(\mathbf{R}^n)$. Hence by (E′) $\langle f(c), \nu(x)\rangle \leq 0$ for $a \leq c \leq b$ and $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma(v)$ where $\nu(x)$ is the normal to $\Gamma(v)$ at x. Consider the convex set

$$\Omega = \left\{ \sum_{i=1}^{m} a_i f(c_i) : a_i \geq 0,\ a \leq c_i \leq b,\ m < \infty \right\}.$$

We have $\nu(x) \in \bigcap_{z\in\Omega} \{w \in \mathbf{R}^n : \langle w, z\rangle \leq 0\}$ for each such $x \in \Gamma(v)$. From (3.1) the interior of Ω is nonempty. Choose $\alpha \in \Omega^0$ with $|\alpha| = 1$ and $\delta > 0$ so that

$$E \equiv \{z \in \mathbf{R}^n : A(z,\alpha) \leq \delta\} \subset \Omega.$$

Then

$$\nu(x) \in \bigcap_{z\in E} \{w \in \mathbf{R}^n : \langle w, z\rangle \leq 0\} = \{w \in \mathbf{R}^n : A(-\alpha, w) \leq \pi/2 - \delta\}.$$

It follows from the theory of sets with locally finite perimeter [**3**, Theorem 4.8] that in a rotated system of coordinates (with $e_n = -\alpha$) there exists a function $g(\hat{y})$ such that $M = \{(\hat{y}, y_n): y_n < g(\hat{y})\}$ where $v = b \cdot \chi_M + a \cdot \chi_{\mathbf{R}^n - M}$ and

$$|g(\hat{y}_1) - g(\hat{y}_2)| \leq \tan(\pi/2 - \delta) \cdot |\hat{y}_1 - \hat{y}_2| \equiv C_0 \cdot |\hat{y}_1 - \hat{y}_2|.$$

If $u_0 \notin BV(\mathbf{R}^n)$ we take a sequence, $\{u_j(0,x)\}$, of admissible data such that $u_j(0,x) \in BV(\mathbf{R}^n)$ and $u_j(0,x) \to u_0(x)$ in $L^1_{\text{loc}}(\mathbf{R}^n)$ as $j \to \infty$. We have seen that the corresponding steady state limits satisfy $v_j \to v$ in $L^1_{\text{loc}}(\mathbf{R}^n)$ as $j \to \infty$. From the above argument we have $M_j = \{(\hat{y}, y_n): y_n < g_j(\hat{y})\}$ where the g_j are uniformly Lipschitz continuous and

$$\langle f(c), (-\nabla g_j(\hat{y}), 1)\rangle \leq 0 \quad \text{for } a \leq c \leq b$$

and L^{n-1} a.e. $\hat{y}$. (Here it is understood that $f(c)$ is expressed in the new coordinates, $(\hat{y}, y_n)$.) It follows that $M = \{(\hat{y}, y_n): y_n < g(\hat{y})\}$ for some Lipschitz continuous g satisfying the same conditions.

The uniform bound on the Lipschitz constant of the shock front yields the following stability result.

THEOREM 3.3. *Let* $u_1(0,x)$, $u_2(0,x)$ *be admissible data and* $v_1(x-kt)$, $v_2(x-kt)$ *be the asymptotic limits with* g_1, g_2 *the corresponding shock fronts. There is a constant* $C_1(\tilde{f}) < \infty$ *so that*

$$||g_1 - g_2||_{L^\infty(\mathbf{R}^{n-1})} \leq C_1 \left(\int_{B_{R_1}} |u_1(0,x) - u_2(0,x)|dx \right)^{1/n}.$$

PROOF. Assume that $f(a) = f(b) = 0$. In the coordinates $(\hat{y}, y_n)$ we have $M_i = \{(\hat{y}, y_n): y_n < g_i(\hat{y})\}$ and $|\nabla g_i| \leq C_0$ for $i = 1, 2$ where $v_i = b \cdot \chi_{M_i} + a \cdot \chi_{\mathbf{R}^n - M_i}$. As in the proof of Theorem 2.4, Green's theorem implies that

$$\begin{aligned}(b-a) \cdot \int_{\mathbf{R}^{n-1}} |g_1 - g_2| d\hat{y} &= \int_{B_{R_3}} |v_1 - v_2| dx \\ &\leq \int_{B_{R_1}} |u_1(0,x) - u_2(0,x)| dx.\end{aligned}$$

But since $|\nabla g_1|$ and $|\nabla g_2|$ are a priori bounded, we conclude that if $|g_1(\hat{y}_0) - g_2(\hat{y}_0)| = \delta > 0$ then $|g_1(\hat{y}) - g_2(\hat{y})| \geq \delta/2$ for $\hat{y} \in B_{\delta/4C_0}(\hat{y}_0)$. Thus

$$(||g_1 - g_2||_{L^\infty(\mathbf{R}^{n-1})})^n \leq C_1(\tilde{f}) \cdot \int_{\mathbf{R}^{n-1}} |g_1 - g_2| d\hat{y}.$$

REMARK 3.4. We will need to apply a version of this theorem when $u_1(0,x)$ and $u_2(0,x)$ are admissible relative to shocks that differ by a small translation, i.e. $u_1(0,x) = \varphi(x)$ and $u_2(0,x) = \varphi(x + \eta\mu)$ for $|x| \geq R_1$. If we assume w.l.o.g. that $C_0 > 1$ and take $R_4 = 5C_0 \cdot \max\{R_3, C_0\eta\}$ then the same argument yields

$$||g_1 - g_2||_{L^\infty(\mathbf{R}^{n-1})} \leq C_1 \left(\int_{B_{R_4}} |u_1(0,x) - u_2(0,x)|dx \right)^{1/n}.$$

DEFINITION 3.5. A function $h(x)$ is *piecewise continuous* iff there exists a finite number of mutually disjoint domains, $\{D_i\}_{i=1}^k$, such that $h|_{D_i}$ has a continuous extension to $D_i \bigcup \partial D_i$ and $L^n \left(\mathbf{R}^n - \bigcup_{i=1}^k D_i \right) = 0$.

We will prove that if u_0 is admissible, piecewise continuous, and $\operatorname{dist}(x - kt, \partial M) > \varepsilon > 0$, then $u(t,x) = v(x-kt)$ after a finite time depending on ε and $\tilde{f}$. (See Theorem 3.12.) It will be convenient to use the following notation.

DEFINITION 3.6. If

$$w(t,x) \in C([0,\infty); L^1_{\text{loc}}(\mathbf{R}^n))$$

we denote by $U(w)(t,x)$ $(L(w)(t,x))$ the upper (lower) Lebesgue limit in x:

$$U(w)(t,x) = \overline{\lim_{r \to 0}} ⨍_{B_r(x)} w(t,z)dz, \quad L(w)(t,x) = \underline{\lim_{r \to 0}} ⨍_{B_r(x)} w(t,z)dz.$$

Note that for each $t > 0$, $U(w)(t,x) = L(w)(t,x) = w(t,x)$ for a.e. x.

The following lemma implies the uniform convergence of $u(t,x)$ away from the shock front.

LEMMA 3.7. *Suppose that u_0 is admissible, piecewise continuous, and $g(\hat{y})$ is the shock front for the asymptotic limit. Then given ε, $\sigma > 0$ there is a constant $T(\varepsilon, \sigma) < \infty$ so that in the coordinates $(\hat{y}, y_n)$ for $t > T$ we have*

(1) $a \leq u(t,y) \leq a + \sigma$ for a.e. y with $y_n - k_n t \geq g(\hat{y} - \hat{k}t) + \varepsilon$,

(2) $b \geq u(t,y) \geq b - \sigma$ for a.e. y with $y_n - k_n t \leq g(\hat{y} - \hat{k}t) - \varepsilon$, where $(\hat{k}, k_n)$ is the representation of $k = [f(b) - f(a)]/(b - a)$ in the coordinates, $(\hat{y}, y_n)$.

PROOF. Assume $f(a) = f(b) = 0$ so that $k = 0$. Note that from Lemma 2.3 the first inequalities in (1) and (2) hold for $t \geq t^*$.

For any $\delta \in (0,1)$ we can find functions $q_\delta(x)$, $p_\delta(x)$ and a positive constant, $\tau(\delta) \leq \delta$, so that the following three properties hold. First,

$$\text{ess inf}\, u_0 \leq q_\delta(x) \leq u_0(x + h) \leq p_\delta(x) \leq \text{ess sup}\, u_0$$

for any $h \in \mathbf{R}^n$ with $|h| \leq \tau$. Second, if $q_\delta(t,x)$ and $p_\delta(t,x)$ are the solutions of (1.1) with initial values $p_\delta(x)$ and $q_\delta(x)$, we have $q_\delta(t,x) = \varphi(x + \tau\mu)$ and $p_\delta(t,x) = \varphi(x - \tau\mu)$ for $|x| \geq R$, $t \geq 0$, where R is independent of δ. Third,

$$\int_{B_R} (p_\delta - q_\delta)\,dx \leq \delta.$$

If (1) does not hold there exists a sequence of points (t^m, y^m) in the rotated coordinates, $y = (\hat{y}, y_n)$, with $t^m \to \infty$ and $y^m \to y^0$ as $m \to \infty$ so that

$$U(u)(t^m, y^m) > a + \sigma \quad \text{and} \quad y_n^m \geq g(\hat{y}^m) + \varepsilon.$$

For δ fixed and any $\bar{y} \in B_{\tau/2}(y^0)$ it follows from Remark 1.7 that

$$u(t^m, y + h) \leq p_\delta(t^m, y) \quad \text{for a.e. } y$$

where $h = y^m - \bar{y}$, m is sufficiently large and $p_\delta(t,y)$ denotes the function $p_\delta(t,x)$ expressed in the rotated spacial coordinates. Thus

$$U(u)(t^m, y^m) \leq U(p_\delta)(t^m, \bar{y}).$$

As a result,

$$a + \sigma \leq p_\delta(t^m, \bar{y}) \quad \text{for a.e. } \bar{y} \in B_{\tau/2}(y^0) \tag{3}$$

if m is large. Let $v_\delta(x)$ be the asymptotic limit of $p_\delta(t,x)$,

$$v_\delta(x) = b \cdot \chi_{M_\delta}(x) + a \cdot \chi_{\mathbf{R}^n - M_\delta}(x).$$

It follows from (3) and Theorem 2.4 that $B_{\tau/2}(y^0) \subset M_\delta$. Using $g_\delta(\hat{y})$ to denote the shock front for v_δ we get $\varepsilon \leq |g_\delta(\hat{y}^0) - g(\hat{y}^0)|$. But from (3.4)

$$||g - g_\delta||_{L^\infty(\mathbf{R}^{n-1})} \leq C\left(\int_{B_{R_4}} |u_0(x) - p_\delta(0,x)|\,dx\right)^{1/n} \leq C \cdot \delta^{1/n}.$$

Since δ can be chosen arbitrarily small this is a contradiction. The verification of (2) is a similar argument using $q_\delta(x)$.

We shall improve on this estimate by using super and subsolutions. The construction of the supersolution is done below in detail.

LEMMA 3.8. *Let $f(c)$ be expressed in the coordinates $(\hat{y}, y_n)$ and suppose $f(a) = f(b) = 0$. Let $g(\hat{y})$ be any Lipschitz continuous function so that for almost every $\hat{y}$,*

$$\begin{aligned}
&\frac{\langle(\nabla g, -1), f(c)\rangle}{c-a} \geq m > 0 \quad \text{for } a \leq c \leq \frac{a+b}{2}, \\
&\langle(\nabla g, -1), f(c)\rangle \geq 0 \qquad\quad \text{for } a \leq c \leq b.
\end{aligned} \tag{3.9}$$

Suppose η and δ are positive constants. Define

$$w(t, \hat{y}, y_n) = \begin{cases} b & \text{for } y_n \leq g(\hat{y}) - \delta + \eta t, \\ (a+b)/2 & \text{for } g(\hat{y}) - \delta + \eta t \leq y_n \leq g(\hat{y}) - mt, \\ a & \text{for } g(\hat{y}) - mt \leq y_n, \end{cases}$$

when $0 \leq t \leq \delta/(m+\eta)$;

$$w(t, \hat{y}, y_n) = \begin{cases} b & \text{for } y_n < g(\hat{y}) - m\delta/(m+\eta), \\ a & \text{for } y_n \geq g(\hat{y}) - m\delta/(m+\eta) \end{cases}$$

when $t \geq \delta/(m+\eta)$.

Then w is a supersolution of (1.1) provided η is sufficiently large depending only on f and the Lipschitz constant of g.

PROOF. Since w is piecewise constant,

$$\partial_t w + \operatorname{div}_x f(w) = \langle|[(w^+, f(w^+)) - (w^-, f(w^-))], \nu(t,y)\rangle d\mathcal{H}^n|_{\Gamma(w)}.$$

We must verify that

$$\langle[(w^+, f(w^+)) - (c, f(c))], \nu\rangle \geq 0 \quad \text{for } w^+ \leq c \leq w^-$$

and $\mathcal{H}^n$ a.e. $(t,y) \in \Gamma(w)$. The jump between the states b and $(a+b)/2$ is determined by

$$0 = y_n - g(\hat{y}) + \delta - \eta t.$$

We have

$$\nu = (-\eta, -\nabla g, 1)/\sqrt{\eta^2 + |\nabla g|^2 + 1}.$$

Thus we need to establish that

$$-\left(\frac{a+b}{2} - c\right)\eta + \left\langle \left[f\left(\frac{a+b}{2}\right) - f(c)\right], (-\nabla g, 1)\right\rangle \geq 0 \quad \text{for } \frac{a+b}{2} \leq c \leq b.$$

This is equivalent to

$$\eta \geq \left\langle \left[f\left(\frac{a+b}{2}\right) - f(c)\right], (-\nabla g, 1)\right\rangle \Big/ \left(\frac{a+b}{2} - c\right)$$

which is valid for η sufficiently large. For the jump between $(a+b)/2$ and a we have $0 = y_n - g(\hat{y}) + mt$, $\nu = (m, -\nabla g, 1)/\sqrt{m^2 + |\nabla g|^2 + 1}$. We must verify that

$$m(a-c) - \langle f(c), (-\nabla g, 1)\rangle \geq 0 \quad \text{for } a \leq c \leq (a+b)/2.$$

This is equivalent to the first inequality of (3.9).

On the jump between b and a for $t \geq \delta/(m+\eta)$ the entropy condition is equivalent to the second inequality of (3.9). This completes the proof.

The shock front g satisfies

$$g(\hat{y}) = \langle l, \hat{y}\rangle \quad \text{for } |\hat{y}| \geq R_3$$

where $(l,-1)/\sqrt{|l|^2+1}$ is the representation of $-\mu$ in the coordinates $(\hat{y}, y_n)$. Hence

$$\langle f(c),(l,-1)\rangle \geq \theta(c-a)(b-c)\sqrt{|l|^2+1} \quad \text{for } a \leq c \leq b.$$

In general the first inequality of (3.9) does not hold. However, g can be uniformly approximated from above and below by functions, $g_{1,\varepsilon}$ and $g_{2,\varepsilon}$ respectively, where

$$g_{j,\varepsilon}(\hat{y}) = g((1-\varepsilon)\hat{y}) + \varepsilon\langle l, \hat{y}\rangle - (-1)^j \cdot c\varepsilon$$

for $0 < \varepsilon < \frac{1}{2}$, $j = 1, 2$, and $c > 0$ is sufficiently large (independent of ε). We have

$$\nabla g_{j,\varepsilon}(\hat{y}) = (1-\varepsilon)\cdot\nabla g((1-\varepsilon)\hat{y}) + \varepsilon l$$

so that

$$\begin{aligned} &\langle f(c),(\nabla g_{j,\varepsilon}(\hat{y}),-1)\rangle \\ (3.10)\qquad &= (1-\varepsilon)\langle f(c),(\nabla g((1-\varepsilon)\hat{y}),-1)\rangle + \varepsilon\langle f(c),(l,-1)\rangle \\ &\geq \varepsilon\theta\sqrt{|l|^2+1}\cdot(c-a)(b-c) \quad \text{for } a \leq c \leq b. \end{aligned}$$

Thus supersolutions can be constructed with $g_{1,\varepsilon}$.

Similarly one can construct subsolutions with $g_{2,\varepsilon}$. The inequalities analogous to (3.9) which we require are

$$\frac{\langle(\nabla g,-1),f(c)\rangle}{b-c} \geq \eta > 0 \quad \text{for } \frac{a+b}{2} \leq c \leq b,$$

$$\langle(\nabla g,-1),f(c)\rangle \geq 0 \quad \text{for } a \leq c \leq b.$$

These hold with g replaced by $g_{2,\varepsilon}$ and η sufficiently small by inequality (3.10). Thus the formula of Lemma 3.8 defines a subsolution if m is sufficiently large.

We now improve on Lemma 3.7 in the sense that we show $u(t,x) = v(x-kt)$ outside of an arbitrarily small neighborhood of the shock front after finite time.

THEOREM 3.11. *Let $u(t,x)$ be a solution of* (1.1) *with $u_0(x)$ admissible and piecewise continuous. Then in the coordinates $(\hat{y}, y_n)$, given $\delta > 0$ there is a constant $T(\delta) < \infty$ so that*

(1) $\quad u(t,y) = a$ *for a.e. y with* $y_n - k_n t \geq g(\hat{y} - \hat{k}t) + \delta,\ t \geq T$,

(2) $\quad u(t,y) = b$ *for a.e. y with* $y_n - k_n t \leq g(\hat{y} - \hat{k}t) - \delta,\ t \geq T$.

PROOF. Assume that $f(a) = f(b) = 0$. Suppose there is a sequence (t^j, y^j) with $y^j \to y^0$, $t^j \to \infty$ as $j \to \infty$ such that $U(u)(t^j, y^j) > a$ and $y_n^0 \geq g(\hat{y}^0) + \delta$. Suppose $\varepsilon \ll \delta$ and set

$$h_\tau(\hat{y}) = g_{1,\varepsilon}(\hat{y}) + \tau, \quad M_\tau = \{(\hat{y}, y_n): y_n < h_\tau(\hat{y})\}.$$

Define

$$\tau(t) = \inf\{\tau: \{y: U(u)(t,y) > a\} \subset M_\tau\}.$$

This is well defined for each $t > 0$ by Theorem 2.4. Set $v_\tau = b\cdot\chi_{M_\tau} + a\cdot\chi_{\mathbf{R}^n - M_\tau}$. By (3.10), v_τ is a steady state solution of (1.1). From Remark 1.7 and Theorem 2.3 we get $v_{\tau(t)}(y) \geq u(s,y)$ for $s \geq t$ and a.e. y. Thus $\tau(t)$ is nonincreasing as $t \uparrow \infty$.

Set $\tau_0 = \lim_{t\to\infty}\tau(t)$. If ε is sufficiently small $y_n^0 \geq h_{3\delta/4}(\hat{y}^0)$, so $\tau_0 \geq 3\delta/4$. Hence $h_{\tau_0} \geq g + 3\delta/4$. Choose T so large that $\tau(T) \leq \varepsilon^2 + \tau_0$; using Lemma 3.7, assume T is large enough to ensure that

$$U(u)(t,y) \leq \frac{a+b}{2} \quad \text{for } y_n \geq g(\hat{y}) + \frac{\delta}{2},\ t \geq T.$$

We can use the supersolution from the previous lemma for $t \geq T$ with g replaced by $h_{\tau(T)}$, δ replaced by $\delta/4$, $m = \varepsilon\theta(b-a)(\sqrt{|l|^2+1})/2 \equiv c_1\varepsilon$, and

$$w(T,y) = \begin{cases} b & \text{for } y_n \leq h_{\tau(T)}(\hat{y}) - \delta/4, \\ (a+b)/2 & \text{for } h_{\tau(T)}(\hat{y}) - \delta/4 \leq y_n \leq h_{\tau(T)}(\hat{y}), \\ a & \text{for } h_{\tau(T)}(\hat{y}) < y_n. \end{cases}$$

It follows that for $t \geq T + \delta/4(\eta + c_1\varepsilon) \equiv T + c_2\delta$, $U(u)(t,y) \equiv a$ on the set $\{y: y_n > h_{\tau(T)}(\hat{y}) - c_1c_2\delta\varepsilon\}$. Thus $\tau_0 \leq \tau(T) - c_1c_2\delta\varepsilon$ which implies that $c_1c_2\delta\varepsilon \leq \varepsilon^2$. This is a contradiction since ε can be taken arbitrarily small. The proof of (1) is complete; (2) follows in a similar way using subsolutions.

An immediate consequence of this theorem is

THEOREM 3.12. *Suppose u_0 is admissible, piecewise continuous, and $k = [f(b) - f(a)]/(b-a)$. Then for any $\varepsilon > 0$ there is a constant $T_\varepsilon < \infty$ such that*

$$u(t,x) = v(x - kt) \quad \textit{for a.e. } x \in \mathbf{R}^n,$$

when $t > T_\varepsilon$ and $\operatorname{dist}(x - kt, \partial M) > \varepsilon$.

4. The lower dimensional case. We now analyze the situation when $d < n$. By an appropriate rotation of the spacial coordinates we can assume that

$$f(c) = (h_1(c), \ldots, h_d(c), j_1(c), \ldots, j_{n-d}(c)),$$

dim span$\{\tilde{h}(c): a \leq c \leq b\} = d$, and $\tilde{j}(c) = 0$ for $a \leq c \leq b$, where

$$\tilde{h}(c) \equiv h(c) - h(a) - (c-a)[h(b) - h(a)]/(b-a),$$
$$\tilde{j}(c) \equiv j(c) - j(a) - (c-a)[j(b) - j(a)]/(b-a).$$

Moreover the projection of μ on $\mathbf{R}^d$ is nonzero and if $\mu' = P_{\mathbf{R}^d}(\mu)/||P_{\mathbf{R}^d}(\mu)||$ we have

$$\langle \mu', \tilde{h}(c)\rangle \leq \langle \mu, \tilde{f}(c)\rangle \leq -\theta(c-a)(b-c) \quad \text{for } a \leq c \leq b.$$

We use the notation $x = (x', x'')$ where $x' \in \mathbf{R}^d$ and $x'' \in \mathbf{R}^{n-d}$.

LEMMA 4.1. *Suppose u_0 is admissible and $k = (k', k'') = [f(b) - f(a)]/(b-a)$. Then for $t \geq T$ (where T is such that $a \leq u(t,x) \leq b$ for $t \geq T$) we have $u(t,x) = \tilde{u}(t, x' - k't; x'' - k''t)$, where $\tilde{u}(t,x';x'') \in C([T,\infty); L^1_{\text{loc}}(\mathbf{R}^d))$ for L^{n-d} a.e. x'' and is the solution of the Cauchy problem*

$$\begin{cases} \partial_t w + \operatorname{div}_{x'} \tilde{h}(w) = 0 & \textit{for } t \geq T,\ x' \in \mathbf{R}^d, \\ w(T,x') = u(T, x' + k'T; x'' + k''T). \end{cases} \tag{4.2}$$

Moreover for each such x'', $u(T, x' + k'T; x'' + k''T)$ is admissible relative to a nondegenerate shock of the form

$$\varphi(x';x'') = \begin{cases} a & \textit{for } \langle x' - x_0'(x''), \mu'\rangle > 0, \\ b & \textit{for } \langle x' - x_0'(x''), \mu'\rangle < 0. \end{cases}$$

PROOF. As in §2 we assume that $f(a) = f(b) = 0$ so that $\tilde{h}(c) = h(c)$, $j(c) = 0$ for $a \leq c \leq b$, and $k = 0$.

It suffices to prove the lemma in the region $\Omega_N = \{(t,x): T \leq t < T + N,\ |x| \leq N\}$ for any $N < \infty$. From (1.3) we see that u is uniquely determined on Ω_N by $u(T,x)$

with $|x| \le N(1+l)$. Thus without loss of generality we can assume that the solution is redefined at time $t = T$ for $|x| > N(1+l)$ so that $u(T,x) \equiv a$ for $|x| \ge 2N(1+l)$.

To begin with suppose $u(T,x) \in C^\infty(\mathbf{R}^n)$ and $h \in C^\infty(\mathbf{R}^1; \mathbf{R}^d)$. For $\varepsilon > 0$ let $v_\varepsilon(t, x'; x'')$ be the solution to

$$\begin{cases} \partial_t v_\varepsilon + \operatorname{div}_{x'} h(v_\varepsilon) = \varepsilon \Delta_{x'} v_\varepsilon & \text{for } t \ge T \text{ and } x \in \mathbf{R}^n, \\ v_\varepsilon(T, x'; x'') = u(T,x) & \text{for } x \in \mathbf{R}^n. \end{cases} \tag{4.3}$$

Thus we view x'' as a parameter in the initial conditions. The theory of parabolic equations implies that a unique solution exists with $a \le v_\varepsilon \le b$ and

$$v_\varepsilon \in C^\infty([T,\infty) \times \mathbf{R}^n).$$

From [**8**, §§17.2 and 18.1], we obtain the estimates

$$\text{(a)} \qquad \int_{\mathbf{R}^d} |\nabla_{x'} v_\varepsilon|(t, x'; x'')\,dx' \le \int_{\mathbf{R}^d} |\nabla_x v_\varepsilon|(T, x'; x'')\,dx' \quad \text{for } t \ge T,\ x'' \in \mathbf{R}^{n-d},$$

$$\text{(b)} \qquad \int_T^{T'} \int_{\mathbf{R}^d} |\partial_t v_\varepsilon|(t, x'; x'')\,dx'\,dt \le c(T', l) \cdot \int_{\mathbf{R}^d} (|\nabla_{x'} v_\varepsilon|(T, x'; x'') + \varepsilon |\Delta_{x'} v_\varepsilon|(T, x'; x''))\,dx'$$

for $T \le T' < \infty$ and $x'' \in \mathbf{R}^{n-d}$. As a result we get

$$\text{(c)} \qquad \int_{\mathbf{R}^n} |\nabla_x v_\varepsilon|(t,x)\,dx \le \int_{\mathbf{R}^n} |\nabla_x v_\varepsilon|(T,x)\,dx \quad \text{for } t \ge T,$$

$$\text{(d)} \qquad \int_T^{T'} \int_{\mathbf{R}^n} |\partial_t v_\varepsilon|(t,x)\,dx\,dt \le C(T', l) \cdot \int_{\mathbf{R}^n} (|\nabla_x v_\varepsilon|(T,x) + \varepsilon |\Delta_{x'} v_\varepsilon|(T,x))\,dx.$$

If we assume only that $h \in C^1([a,b]; \mathbf{R}^d)$ then by taking a sequence of smooth functions, h_j, converging to h in this space one finds that the corresponding solutions, $v_{\varepsilon,j}$, together with $\nabla_x v_{\varepsilon,j}$ converge uniformly in $[T,T'] \times \mathbf{R}^n$. Also

$$\sup_{[T,T'] \times R^n} |\partial_t v_{\varepsilon,j}| \le C < \infty$$

independent of j. Letting $j \to \infty$ the limit, v_ε, will be a solution of (4.3); the integrands in (a)–(d) are all locally integrable functions and since the right-hand sides of the four inequalities are independent of j they remain valid in the limit.

From (c) and (d) we see that a sequence $\varepsilon_j \downarrow 0$ can be chosen so that $v_{\varepsilon_j} \to w \in BV((T,\infty) \times \mathbf{R}^n)$ with convergence in $L^1_{\text{loc}}((T,\infty) \times \mathbf{R}^n)$. We can further assume that there is a set, Z, of full $\mathcal{L}^{n-d}$ measure so that

$$v_{\varepsilon_j}(t,x) \to w(t, x'; x'') \quad \text{in } L^1_{\text{loc}}((T,\infty) \times \mathbf{R}^d)$$

for $x'' \in Z$. From (a) and (b) we see that $w(t,x';x'') \in BV((T,\infty) \times \mathbf{R}^d)$ for each $x'' \in Z$. Thus (as in §1) we can define

$$\bar{w}(t, x'; x'') \equiv \lim_{r \to \infty} \fint_{B_r((t,x'))} w(\tau, z; x'')\,d\tau\,dz, \qquad t > T,\ x'' \in Z.$$

From [**8**, §18], for each such x'', $\bar{w}(t,x';x'')$ is the solution to (4.2). By the same argument as in [**8**, §18], $\bar{w}(t,x)$ (the Lebesgue limit in $\mathbf{R}^{n+1}$) is the solution to (1.1) with $\bar{w}(t,x) \to u(T,x)$ as $t \downarrow T$ in $L^1_{\text{loc}}(\mathbf{R}^n)$. We point out that $\bar{w}(t,x';x'')$ agrees with $\bar{w}(t,x)$ as an element in $C([T,\infty);L^1_{\text{loc}}(\mathbf{R}^n))$. We point out that $\bar{w}(t,x';x'')$ agrees with $\bar{w}(t,x)$ as an element in $C([T,\infty);L^1_{\text{loc}}(\mathbf{R}^n))$. Thus the proof is complete when $u(T,x) \in C^\infty(\mathbf{R}^n)$.

For the general case consider a sequence of functions $u_j(T,x) \in C^\infty(\mathbf{R}^n)$ with $a \le u_j \le b$, $u_j(T,x) = a$ for $|x| \ge 2N(1+l)$, and $u_j \to u$ in $L^1_{\text{loc}}(\mathbf{R}^n)$. Using equation (1.3) we see that $\{\bar{u}_j(t,x';x'')\}$, $j = 1,2,\ldots$, forms a Cauchy sequence in $C([T,T+N];L^1_{\text{loc}}(\mathbf{R}^n))$, and that for a.e. $x'' \in \mathbf{R}^{n-d}$, the sequence is Cauchy in $C([T,T+N];L^1_{\text{loc}}(\mathbf{R}^d))$. For a subsequence, $\bar{u}_j(t,x';x'') \to u(t,x)$ for all $t \in [T,T+N]$ and a.e. $x'' \in \mathbf{R}^{n-d}$, pointwise almost everywhere in $\mathbf{R}^d$. This limit has the required properties.

With this result and those from §§2 and 3 we have the following.

THEOREM 4.4. *Let $u_0(x)$ be admissible and let $b\cdot\chi_M(x-kt)+a\cdot\chi_{\mathbf{R}^n-M}(x-kt)$ be the asymptotic limit of $u(t,x)$. Then for almost every $x_0'' \in \mathbf{R}^{n-d}$, $M(x_0'') \equiv \{x' : (x',x_0'') \in M\}$ is a Lipschitz domain in $\mathbf{R}^d$. Moreover there is a rotation of the x' coordinates to $y' = (\hat{y}, y_d) \in \mathbf{R}^{d-1} \times \mathbf{R}^1$ and a function $g(\hat{y},x'')$ defined for almost every x'', Lipschitz continuous in $\hat{y}$ (with Lipschitz constant independent of x''), so that*

$$M(x'') = \{y' \in \mathbf{R}^d : y_d < g(\hat{y},x'')\}.$$

LEMMA 4.5. *Suppose $u_0(x)$ is admissible and piecewise continuous. Let $\tilde{u}$ be as in Lemma 4.1 but with x' replaced by the rotated coordinates, $y' = (\hat{y}, y_d)$. Then for almost every $x'' \in \mathbf{R}^{n-d}$ and any $\delta > 0$ there is a constant $T(\delta,x'') < \infty$ so that*

$$\tilde{u}(t,y';x'') = a \quad \textit{for a.e. } y \textit{ such that} \quad y_n \ge g(\hat{y},x'') + \delta,\ t \ge T,$$

$$\tilde{u}(t,y';x'') = b \quad \textit{for a.e. } y \textit{ such that} \quad y_n \le g(\hat{y},x'') - \delta,\ t \ge T.$$

PROOF. Assume that $f(a) = f(b) = 0$, so that $k = 0$ and $u = \tilde{u}$. Consider for each $j = 1,2,\ldots$, $q_{1/j}$ and $p_{1/j}$ as defined in Lemma 3.7. For $t \ge t^*$, using Theorem 2.3,

$$a \le q_{1/j}(t,x) \le p_{1/j}(t,x) \le b.$$

Now there exists a set Z of full $\mathcal{L}^{n-d}$ measure so that for all j,

$$q_{1/j}(t^*,x',x'') \le u(t^*,x'+h,x'') \le p_{1/j}(t^*,x',x'')$$

for $\mathcal{L}^d$ a.e. x', for each $|h| < \tau(1/j)$ and $x'' \in Z$. By construction $p_{1/j}(t^*,x) - q_{1/j}(t^*,x) \to 0$ as $j \to \infty$ in $L^1_{\text{loc}}(\mathbf{R}^n)$. Hence for almost every $x'' \in Z$,

$$\int_{|x'|\le R} [p_{1/j}(t^*,x',x'') - q_{1/j}(t^*,x',x'')]dx' \to 0$$

as $j \to \infty$ for any $R < \infty$. Using Lemma 4.1 we can argue as in Lemma 3.7 and Theorem 3.11 for each such x'' to obtain the conclusion of the lemma.

LEMMA 4.6. *Let $u_0(x)$ be admissible and piecewise continuous. Then for all $\varepsilon > 0$ and almost every $z'' \in \mathbf{R}^{n-d}$ there is a constant, $T(\varepsilon,z'')$, such that if*

$$x'' - k''t = z'',$$

$$u(t, x', x'') = \tilde{u}(t, x' - k't; x'' - k''t)$$
$$= b \cdot \chi_{M(x''-k''t)}(x' - k't) + a \cdot \chi_{R^d - M(x''-k''t)}(x' - k't)$$

for $t \geq T$ *and almost every* x' *such that* $\operatorname{dist}(x' - k't, \partial M(x'' - k''t)) > \varepsilon$.

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